Choosability in signed planar graphs

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Abstract

This paper studies the choosability of signed planar graphs. We prove that every signed planar graph is 5-choosable and that there is a signed planar graph which is not 4-choosable while the unsigned graph is 4-choosable. For each $k \in \{3, 4, 5, 6\}$, every signed planar graph without circuits of length k is 4-choosable. Furthermore, every signed planar graph without circuits of length 3 and of length 4 is 3-choosable. We construct a signed planar graph with girth 4 which is not 3-choosable but the unsigned graph is 3-choosable.

1 Introduction

This paper discusses simple graphs. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We say a vertex u is a *neighbor* of another vertex v if $uv \in E(G)$. If $v \in V(G)$, then $d(v)$ denotes the degree of v and furthermore, v is called a k-*vertex* (or k^+-*vertex* or k^--*vertex*) if $d(v) = k$ (or $d(v) \geq k$ or $d(v) \leq k$). Similarly, a k-*circuit* (or k^+-*circuit* or k^--*circuit*) is a circuit of length k (or at least k or at most k), and if G is planar then a k-*face* (or k^+-*face* or k^--*face*) is a face of size k (or at least k or at most k). Let $[x_1 \dots x_k]$ denote a k-circuit with vertices $x_1, \dots, x_k$ in cyclic order. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, and $\partial(X)$ denotes the set of edges between X and $V(G) \setminus X$.

Let G be a graph and $\sigma : E(G) \to \{1, -1\}$ be a mapping. The pair (G, σ) is called a *signed graph*, and σ is called a *signature* of G. An edge e is *positive* (or *negative*) if $\sigma(e) = 1$ (or $\sigma(e) = -1$). Denote by $(G, +)$ the signed graph (G, σ) with $\sigma(e) = 1$ for each $e \in E(G)$. A graph with no signature is usually called an *unsigned graph*. A circuit of a signed graph is balanced (unbalanced) if it contains an even (odd) number of negative edges.

Zaslavsky [12] defines a (signed) coloring of a signed graph (G, σ) with k colors or with $2k+1$ signed colors to be a mapping $c : V(G) \longrightarrow \{-k, -(k-1), \dots, -1, 0, 1 \dots, (k-1), k\}$

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such that for every edge uv of G, $c(u) \neq c(v)$ if $\sigma(uv) = 1$, and $c(u) \neq -c(v)$ if $\sigma(uv) = -1$. Recently, Máčajová, Raspaud and Škoviera [4] introduced a *k-coloring* of (G, σ) as a proper coloring of (G, σ) using colors from $\{\pm 1, \pm 2, \ldots, \pm \frac{k}{2}\}$ if $k \equiv 0 \pmod 2$, and ones from $\{0, \pm 1, \pm 2, \ldots, \pm \frac{k-1}{2}\}$ if $k \equiv 1 \pmod 2$. A signed graph (G, σ) is *k-colorable* if it admits a k-coloring. The *chromatic number* of (G, σ) is the minimum number k such that (G, σ) is k-colorable. We follow the approach of [4] to define list colorings of signed graphs. Given a signed graph (G, σ), a *list-assignment* of (G, σ) is a function L defined on $V(G)$ such that $\emptyset \neq L(v) \subseteq \mathbb{Z}$ for each $v \in V(G)$. An *L-coloring* of (G, σ) is a proper coloring c of (G, σ) such that $c(v) \in L(v)$ for each $v \in V(G)$. A list-assignment L is called a *k-list-assignment* if $|L(v)| = k$ for each $v \in V(G)$. We say (G, σ) is *k-choosable* if it admits an L-coloring for every k-list-assignment L. The *choice number* of (G, σ) is the minimum number k such that (G, σ) is k-choosable. Clearly, if a signed graph is k-choosable, then it is also k-colorable.

Let (G, σ) be a signed graph, L be a list assignment of (G, σ), and c be an L-coloring of (G, σ). Let $X \subseteq V(G)$. We say σ', L' and c' are obtained from σ, L and c by a *switch* at X if

$$\sigma'(e) = \begin{cases} -\sigma(e), & \text{if } e \in \partial(X), \\ \sigma(e), & \text{if } e \in E(G) \setminus \partial(X), \end{cases} \qquad L'(u) = \begin{cases} \{-\alpha\colon\ \alpha \in L(u)\}, & \text{if } u \in X, \\ L(u), & \text{if } u \in V(G) \setminus X, \end{cases}$$

$$c'(u) = \begin{cases} -c(u), & \text{if } u \in X, \\ c(u), & \text{if } u \in V(G) \setminus X. \end{cases}$$

Two signed graphs (G, σ) and (G, σ^*) are *equivalent* if they can be obtained from each other by a switch at some subset of $V(G)$. Let $\mathcal{G}(G, \sigma) = \{(G, \sigma_1)\colon\ (G, \sigma_1) \text{ is equivalent to } (G, \sigma)\}$.

Proposition 1.1. *Let (G, σ) be a signed graph, L be a list-assignment of G and c be an L-coloring of (G, σ). If σ', L' and c' are obtained from σ, L and c by a switch at a subset of $V(G)$, then c' is an L'-coloring of (G, σ'). Furthermore, two equivalent signed graphs have the same chromatic number and the same choice number.*

Let G be a graph. By definition, G and $(G, +)$ have the same chromatic number and the same choice number. Hence, the following statement holds.

Corollary 1.2. *If $(G, \sigma) \in \mathcal{G}(G, +)$, then G and (G, σ) have the same chromatic number and the same choice number.*

This paper focusses on the choosability of signed planar graphs and generalizes the results of [2, 3, 5, 6, 7, 11] to signed graphs. Section 2 proves that every signed planar graph is 5-choosable. Furthermore, there is a signed planar graph (G, σ) which is not 4-choosable, but $(G, +)$ is 4-choosable. Section 3 proves for every $k \in \{3, 4, 5, 6\}$ that every

signed planar graph without k-circuits is 4-choosable. Section 4 proves that every signed planar graph with neither 3-circuits nor 4-circuits is 3-choosable. Furthermore, there exists a signed planar graph (G, σ) such that G has girth 4 and (G, σ) is not 3-choosable but $(G, +)$ is 3-choosable.

2 5-choosability

Theorem 2.1. *Every signed planar graph is 5-choosable.*

We use the method described in [5] to prove following theorem which implies Theorem 2.1. A plane graph G is a *near triangulation* if the boundary of each bounded face of G is a triangle.

Theorem 2.2. *Let (G, σ) be a signed graph, where G is a near-triangulation. Let C be the boundary of the unbounded face of G and $C = [v_1 \dots v_p]$. If L is a list-assignment of (G, σ) such that $L(v_1) = \{\alpha\}$, $L(v_2) = \{\beta\}$ and $\alpha \neq \beta\sigma(v_1v_2)$, and that $|L(v)| \geq 3$ for $v \in V(C) \setminus \{v_1, v_2\}$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus V(C)$, then (G, σ) has an L-coloring.*

Proof. Let us prove Theorem 2.2 by induction on $|V(G)|$.

If $|V(G)| = 3$, then $p = 3$ and $G = C$. Choose a color from $L(v_3) \setminus \{\alpha\sigma(v_1v_3), \beta\sigma(v_2v_3)\}$ for v_3. So we proceed to the induction step.

If C has a chord which divides G into two graphs G_1 and G_2, then we choose the notation such that G_1 contains v_1v_2, and we apply the induction hypothesis first to G_1 and then to G_2. Hence, we can assume that C has no chord.

Let $v_1, u_1, u_2, \dots, u_m, v_{p-1}$ be the neighbors of v_p in cyclic order around v_p. Since the boundary of each bounded face of G is a triangle, G contains the path $P\colon v_1u_1 \dots u_mv_{p-1}$. Since C has no chord, $P \cup (C - v_p)$ is a circuit C'. Let γ_1 and γ_2 be two distinct colors of $L(v_p) \setminus \{\alpha\sigma(v_1v_p)\}$. Define $L'(x) = L(x) \setminus \{\gamma_1\sigma(v_px), \gamma_2\sigma(v_px)\}$ for $x \in \{u_1, \dots, u_m\}$, and $L'(x) = L(x)$ for $x \in V(G) \setminus \{v_p, u_1, \dots, u_m\}$. Let σ' be the restriction of σ to $G - v_p$. By the induction hypothesis, signed graph $(G - v_p, \sigma')$ has an L'-coloring. Let c be the color vertex v_{p-1} receives. We choose a color from $\{\gamma_1, \gamma_2\} \setminus \{c\sigma(v_{p-1}v_p)\}$ for v_p, giving an L-coloring of (G, σ). □

non-4-choosable examples

Voigt [9, 10] constructed two planar graphs which are not 4-choosable. By Corollary 1.2 these two examples generate two group of signed planar graphs which are not 4-choosable. We extend this result to signed graphs.

Theorem 2.3. *There exists a signed planar graph (G, σ) such that (G, σ) is not 4-choosable but G is 4-choosable.*

Proof. We construct (G,σ) as follows. Take a copy G_1 of complete graph K_4 and embed it into Euclidean plane. Insert a claw into each 3-face of G_1 and denote the resulting graph by G_2. Once again, insert a claw into each 3-face of G_2 and denote by G_3 the resulting graph. A vertex v of G_3 is called an initial-vertex if $v \in V(G_1)$, a solid-vertex if $v \in V(G_2) \setminus V(G_1)$ and a hollow-vertex if $v \in V(G_3) \setminus V(G_2)$ (Figure 1 illustrates graph G_3). A 3-face of G_3 is called a special 3-face if it contains an initial-vertex, a solid-vertex and a hollow-vertex. Clearly, G_3 has 24 special 3-faces, say $T_1, \ldots, T_{24}$.

Let H be the plane graph as shown in Figure 2, which consists of a circuit $[xyz]$ and its interior. For $i \in \{1, \ldots, 24\}$, replace T_i by a copy H_i of H such that x_i, y_i and z_i are identified with the solid-vertex, hollow-vertex and initial-vertex of T_i, respectively. Let G be the resulting graph. Clearly, G is planar.

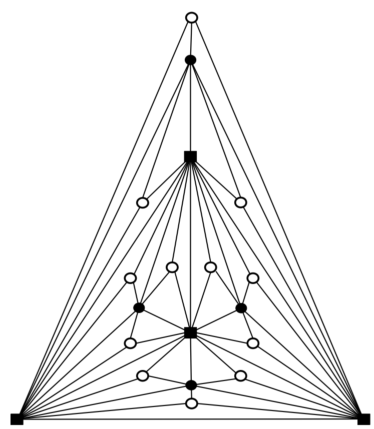

Figure 1: graph G_3

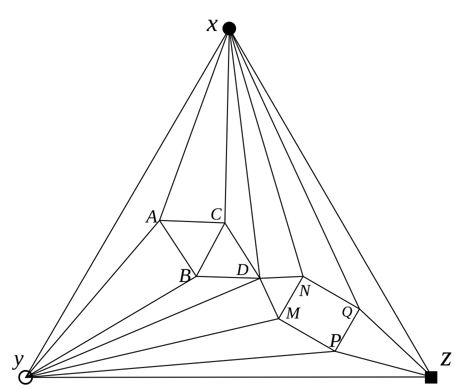

Figure 2: graph H

Define a signature σ of G as follows: $\sigma(P_iQ_i) = -1$ for $i \in \{1, \ldots, 24\}$ and $\sigma(e) = 1$ for $e \in E(G) \setminus \{P_iQ_i : i \in \{1, \ldots, 24\}\}$.

Let L be a 4-list-assignment of signed graph (G,σ) defined as follows: $L(v) = \{1, 2, 3, 4\}$

for $v \in V(G_3)$, and $L(A_i) = \{1, 2, 6, 7\}$, $L(B_i) = \{2, 4, 6, 7\}$, $L(C_i) = \{1, 4, 6, 7\}$, $L(D_i) = \{1, 2, 4, 5\}$, $L(M_i) = \{2, 5, 6, -6\}$, $L(N_i) = \{1, 5, 6, -6\}$, $L(P_i) = \{2, 3, 6, -6\}$ and $L(Q_i) = \{1, 3, 6, -6\}$ for $i \in \{1, \ldots, 24\}$.

We claim that signed graph (G, σ) has no L-coloring. Suppose to the contrary that ϕ is an L-coloring of (G, σ). By the construction of G_3, precisely one of the special 3-faces of G_3 is assigned in ϕ color 1 to its solid-vertex, color 2 to its hollow-vertex and color 3 to its initial-vertex. Without loss of generality, let T_1 be such a special 3-face. Let us consider ϕ in H_1. Clearly, $\phi(x_1) = 1, \phi(y_1) = 2$ and $\phi(z_1) = 3$. It follows that $\phi(D_1) \in \{4, 5\}$. Notice that the odd circuit $[A_1B_1C_1]$ is balanced and the even circuit $[M_1N_1Q_1P_1]$ is unbalanced, and thus both of them are not 2-choosable. It follows that if $\phi(D_1) = 4$, then ϕ is not proper in $[A_1B_1C_1]$, and that if $\phi(D_1) = 5$, then ϕ is not proper in $[M_1N_1Q_1P_1]$. Therefore, (G, σ) has no L-coloring and thus is not 4-choosable.

Let L' be any 4-list-assignment of G. By the construction, it is not hard to see that G_3 is 4-choosable. Let c be an L'-coloring of G_3. Clearly, for $i \in \{1, \ldots, 24\}$, each of vertices x_i, y_i and z_i receives a color in c. Let α and β be two distinct colors from $L(D_i) \setminus \{c(x_i), c(y_i)\}$. Choose a color from $L(C_i) \setminus \{\alpha, \beta, c(x_i)\}$ for C_i, and then vertices A_i, B_i and D_i can be list-colored by L' in turn. Since circuit $[M_iN_iQ_iP_i]$ is 2-choosable, it follows that vertices M_i, N_i, P_i and Q_i can also be list-colored by L'. Therefore, c can be extended to an L'-coloring of G. This completes the proof that G is 4-choosable. $\square$

3 4-choosability

A graph G is *d-degenerate* if every subgraph H of G has a vertex of degree at most d in H. It is known that every $(d-1)$-degenerate graph is d-choosable. This proposition can be extended for signed graphs.

Theorem 3.1. *Let (G, σ) be a signed graph. If G is $(d-1)$-degenerate, then (G, σ) is d-choosable.*

Proof. (induction on $|V(G)|$) Let L be any d-list-assignment of G. The proof is trivial if $|V(G)| = 1$. For $|V(G)| \geq 2$, since G is $(d-1)$-degenerate, G has a vertex v of degree at most $d-1$ and moreover, graph $G - v$ is $(d-1)$-degenerate. Let σ' and L' be the restriction of σ and L to $G - v$, respectively. By applying the induction hypothesis to $(G - v, \sigma')$, we conclude that $(G - v, \sigma')$ is d-choosable and thus has an L'-coloring ϕ. Since v has degree at most $d-1$, we can choose a color α for v such that $\alpha \in L(v) \setminus \{\phi(u)\sigma(uv) \colon\ uv \in E(G)\}$. We complete an L-coloring of (G, σ) with ϕ and α. $\square$

It is an easy consequence of Euler's formula that every triangle-free planar graph contains a vertex of degree at most 3. Therefore, the following statement is true:

Lemma 3.2. *Planar graphs without 3-circuits are 3-degenerate.*

Moreover, we will use two more lemmas.

Lemma 3.3 ([11])**.** *Planar graphs without 5-circuits are 3-degenerate.*

Lemma 3.4 ([2])**.** *Planar graph without 6-circuits are 3-degenerate.*

Theorem 3.5. *Let (G, σ) be a signed planar graph. For all $k \in \{3, 4, 5, 6\}$, if G has no k-circuit, then (G, σ) is 4-choosable.*

Proof. For $k \in \{3, 5, 6\}$ we deduce the statement from Theorem 3.1, together with Lemmas 3.2, 3.3 and 3.4, respecvitely. It remains to prove Theorem 3.5 for the case $k = 4$.

Suppose to the contrary that the statement is not true. Let (G, σ) be a counterexample of smallest order, and L be a 4-list-assignment of (G, σ) such that (G, σ) has no L-coloring. Clearly, G is connected by the minimality of (G, σ).

Claim 3.5.1. *$\delta(G) \geq 4$.*

Let u be a vertex of G of minimal degree. Suppose to the contrary that $d(u) < 4$. Let σ' and L' be the restriction of σ and L to $G - u$, respectively. By the minimality of (G, σ), the signed graph $(G - u, \sigma')$ has an L'-coloring c. Since every neighbor of u forbids one color for u no matter what the signature of the edge between them is, $L(u)$ still has a color left for coloring u. Therefore, c can be extended to an L-coloring of (G, σ), a contradiction.

Claim 3.5.2. *G has no 6-circuit C such that $C = [u_0 \dots u_5]$ and $u_0u_2 \in E(G)$, and $d(u_0) \leq 5$ and all other vertices of C are of degree 4.*

Suppose to the contrary that G has such 6-circuit C. Since G has no 4-circuit, u_0u_2 is the only chord of C. There always exists a subset X of $V(C)$ such that all of the edges u_0u_2, u_1u_2 and u_2u_3 are positive after a switch at X. Let σ' and L' be obtained from σ and L by a switch at X, respectively. Proposition 1.1 implies that signed graph (G, σ') has no L'-coloring. Hence, (G, σ') is also a minimal counterexample. Let σ_1 and L_1 be the restriction of σ' and L' to $G - V(C)$, respectively. It follows that $(G - V(C), \sigma_1)$ has an L_1-coloring ϕ.

We obtain a contradiction by further extending ϕ to an L'-coloring of (G, σ') as follows. By the condition on the vertex degrees of C, there exists a list-assignment L_2 of $G[V(C)]$ such that $L_2(u) \subseteq L'(u) \setminus \{\phi(v)\sigma'(uv)\colon\ uv \in E(G) \textit{ and } v \notin V(C)\}$ for $u \in V(C)$, and $|L_2(u_2)| = 3$ and $|L_2(u)| = 2$ for $u \in V(C) \setminus \{u_2\}$. Let $L_2(u_2) = \{\alpha, \beta, \gamma\}$. Suppose that $L_2(u_2)$ has a color, say α, not appear in at least two of lists $L_2(u_0), L_2(u_1)$ and $L_2(u_3)$. We color u_2 with α, and then all other vertices of C can be list-colored by L_2 in some order. For example, if α does not appear in $L_2(u_0)$ and $L_2(u_1)$, then we color $V(C)$ in the order

$u_2, u_3, u_4, u_5, u_0, u_1$. Hence, we may assume that $L_2(u_0) = \{\alpha, \gamma\}$, $L_2(u_1) = \{\alpha, \beta\}$ and $L_2(u_3) = \{\beta, \gamma\}$. If $\beta \neq \gamma\sigma'(u_0u_1)$, then color u_0 with γ, u_1 with β, and u_2 with α, and the remaining vertices of C can be list-colored by L_2 in the order u_5, u_4, u_3. Hence, we may assume $\beta = \gamma\sigma'(u_0u_1)$. It follows that $\sigma'(u_0u_1) = -1$ and $\beta = -\gamma \neq 0$. If $\alpha \neq 0$, then color both u_0 and u_1 with α, and the remaining vertices of C can be list-colored by L_2 in the order u_5, u_4, u_3, u_2. Hence, we may assume $\alpha = 0$. Now color 0 is included in list $L_2(u_0)$ but no in list $L_2(u_3)$. Thus there exists an integer i in set $\{3, 4, 5\}$ such that $0 \in L_2(u_{i+1})$ and $0 \notin L_2(u_i)$ (index is added modular 6). We color u_{i+1} with color 0, and then the remaining vertices of C can be list-colored by L_2 in cyclic order on C ending at u_i.

Claim 3.5.3. *G has no 10-circuit C such that $C = [u_0 \dots u_9]$ and $u_0u_8, u_2u_6, u_2u_7 \in E(G)$, and vertex u_2 has degree 6 and all other vertices of C have degree 4.*

Suppose to the contrary that G has such a 10-circuit C. Let σ' and L' be the restriction of σ and L to graph $G - V(C)$, respectively. By the minimality of (G, σ), signed graph $(G - V(C), \sigma')$ has an L'-coloring ϕ. A contradiction is obtained by further extending ϕ to an L-coloring of (G, σ) as follows. We shall list-color the vertices of C by L in the cyclic order $u_0, u_1, \dots, u_9$. For $i \in \{0, \dots, 9\}$, let $F_i = \{\phi(v)\sigma(u_iv)\colon\ u_iv \in E(G) \textit{ and } v \notin V(C)\}$. Clearly, F_i is the set of forbidden colors by the neighbors of u_i not on C to be assigned to vertex u_i. Since $d(u_0) = d(u_9) = 4$ and moreover, if there is any other chord of C then the list F_i will not become longer, it follows that $|F_0| \leq 1$ and $|F_9| \leq 2$. Hence, we can let α and β be two distinct colors from $L(u_9) \setminus F_9$, and let $\gamma \in L(u_0) \setminus (F_0 \cup \{\alpha\sigma(u_0u_9), \beta\sigma(u_0u_9)\})$. Color vertex u_0 with γ. For $i \in \{1, \dots, 8\}$, vertex u_i has at most 3 neighbors colored before u_i in this color-assigning process and thus, $L(u_i)$ still has a color available for u_i. Denote by ζ the color vertex u_8 receives. We complete the extending of ϕ by assigning a color from $\{\alpha, \beta\} \setminus \{\zeta\sigma(u_8u_9)\}$ to u_9.

Discharging

Consider an embedding of G into the Euclidean plane. Let G denote the resulting plane graph. We say two faces are *adjacent* if they share an edge. Two adjacent faces are *normally adjacent* if they share an edge xy and no vertex other than x and y. Since G is a simple graph, the boundary of every 3-face or 5-face is a circuit. Since G has no 4-circuits, we can deduce that if a 3-face and a 5-face are adjacent, then they are normally adjacent. A vertex is *bad* if it is of degree 4 and incident with two nonadjacent 3-faces. A *bad 3-face* is a 3-face containing three bad vertices. A 5-face f is *magic* if it is adjacent to five 3-faces, and if all the vertices of these six faces have degree 4 except one vertex of f.

We shall obtain a contradiction by applying discharging method. Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G. Denote by $d(f)$ the size of a face f of G. Give

initial charge $ch(x)$ to each element x of $V \cup F$, where $ch(v) = 3d(v) - 10$ for $v \in V$, and $ch(f) = 2d(f) - 10$ for $f \in F$. Discharge the elements of $V \cup F$ according to the following rules:

R1. Every vertex u sends each incident 3-face charge 1 if u is a bad vertex, and charge 2 otherwise.

R2. Every 5-vertex sends $\frac{1}{3}$ to each incident 5-face.

R3. Every 6-vertex sends each incident 5-face f charge 1 if f is magic, charge $\frac{2}{3}$ if f is not magic but contains four 4-vertices, charge $\frac{1}{3}$ if f contains at most three 4-vertices.

R4. Every 7^+-vertex sends 1 to each incident 5-face.

R5. Every 3-face sends $\frac{1}{3}$ to each adjacent 5-face if this 3-face contains at most one bad vertex.

R6. Every 5^+-face sends $\frac{k}{3}$ to each adjacent bad 3-face, where k is the number of common edges between them.

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ when the discharging process is over. On one hand, by Euler's formula we deduce $\sum_{x \in V \cup F} ch(x) = -20$. Since the sum of charge over all elements of $V \cup F$ is unchanged, we have $\sum_{x \in V \cup F} ch^*(x) = -20$. On the other hand, we show that $ch^*(x) \geq 0$ for $x \in V \cup F$. Hence, this obvious contradiction completes the proof of Theorem 3.5.

It remains to show that $ch^*(x) \geq 0$ for $x \in V \cup F$.

Claim 3.5.4. *If $v \in V$, then $ch^*(v) \geq 0$.*

Let p be the number of 3-faces that contains v. Since G has no 4-circuit, $p \leq \lfloor \frac{d(v)}{2} \rfloor$. Moreover, $d(v) \geq 4$ by Claim 3.5.1.

Suppose $d(v) = 4$. We have $p \leq 2$. If $p = 2$, then v is a bad vertex and thus we have $ch^*(v) = 3d(v) - 10 - p = 0$ by R1; otherwise, we have $ch^*(v) = 3d(v) - 10 - 2p \geq 0$ by R1 again.

If $d(v) = 5$, then $p \leq 2$ and thus by R1 and R2, we have $ch^*(v) \geq 3d(v) - 10 - 2p - \frac{1}{3}(5 - p) \geq 0$.

Suppose that $d(v) = 6$. Thus $p \leq 3$. By R1 and R3, if $p \leq 2$ then we have $ch^*(v) \geq 3d(v) - 10 - 2p - (6 - p) \geq 0$, and if v is incident with no magic 5-face then we have $ch^*(v) \geq 3d(v) - 10 - 2p - \frac{2}{3}(6 - p) \geq 0$. Hence, we may assume that $p = 3$ and that v is incident with a magic 5-face f. For any other 5^+-face f' containing v than f, Claim 3.5.3 implies that if f' has size 5 then it contains at most three 4-vertices, and thus v sends at most $\frac{1}{3}$ to f' by R3. Hence, we have $ch^*(v) \geq 3d(v) - 10 - 2 \times 3 - 1 - \frac{1}{3} \times 2 > 0$.

It remains to suppose $d(v) \geq 7$. By R1 and R4, we have $ch^*(v) \geq 3d(v) - 10 - 2p - (d(v) - p) \geq 2d(v) - 10 - \lfloor \frac{d(v)}{2} \rfloor > 0$.

Claim 3.5.5. *If $f \in F$, then $ch^*(f) \geq 0$.*

Suppose $d(f) = 3$. Recall that in this case the boundary of f is a circuit. We have $ch^*(f) \geq 2d(f) - 10 + 2 + 2 + 1 - 3 \times \frac{1}{3} = 0$ by R1 and R5 when f has at most one bad vertex, and $ch^*(f) \geq 2d(f) - 10 + 2 + 1 + 1 = 0$ by R1 when f has precisely two bad vertices. It remains to assume that f has precisely three bad vertices, that is, f is a bad 3-face. In this case, f receives charge 1 in total from adjacent faces by R6, and charge 3 in total from incident vertices by R1. Hence, we have $ch^*(f) \geq 2d(f) - 10 + 1 + 3 = 0$.

Suppose $d(f) = 5$. Recall in this case that the boundary of f is a circuit and that if f is adjacent to a 3-face then they are normally adjacent. Let q be the number of bad 3-faces adjacent to f. Clearly, f sends charge only to adjacent bad 3-faces by R6, and possibly receives charge from incident 5^+-vertices and adjacent 3-faces by rules from R2 to R5. Hence, we have $ch^*(f) \geq 2d(f) - 10 = 0$ when $q = 0$. Claim 3.5.2 implies that $q \leq 3$ and that f contains a 5^+-vertex u, which sends at least $\frac{1}{3}$ to f. Hence, we have $ch^*(f) \geq 2d(f) - 10 - \frac{1}{3} + \frac{1}{3} = 0$ when $q = 1$. First suppose $q = 2$. If f has a 5^+-vertices different from u, then we are done by $ch^*(f) \geq 2d(f) - 10 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0$. Hence, we may assume that f contains four 4-vertices. It follows that if $d(u) \geq 6$, then f receives at least $\frac{2}{3}$ from v by R3 or R4 and thus we are done. Hence, we may assume that $d(u) = 5$. Through the drawing of 3-faces adjacent to f, we can assume u is incident with a 3-face $[uvw]$ that is adjacent to f on edge uv. Claim 3.5.2 implies that $d(w) \geq 5$. Hence, f receives $\frac{1}{3}$ from face $[uvw]$ by R5, and thus we are done. Let us next suppose $q = 3$. We may assume $f = [uv'w'x'y']$ such that $v'w'$, $w'x'$ and $x'y'$ are the three common edges between f and bad 3-faces. Since both vertices v' and y' are bad, edges uv' and uy' are contained in 3-faces $[uv't']$ and $[uy'z']$, respectively. If $d(u) = 5$, then Claim 3.5.2 implies that $d(t'), d(z') \geq 5$, and thus f receives $\frac{1}{3}$ from each of faces $[uv't']$ and $[uy'z']$ by R5, we are done. If $d(u) \geq 7$, then f receives 1 from u and thus we are done. Hence, we may assume that $d(u) = 6$. If both t' and z' has degree 4, that is, f is a magic 5-face, then f receives 1 from u by R3; otherwise, f receives $\frac{2}{3}$ from u and $\frac{1}{3}$ from at least one of faces $[uv't']$ and $[uy'z']$ by R3 again. We are done in both cases.

It remains to suppose $d(f) \geq 6$. Remind that f has no charge moving in or out except that it sends $\frac{1}{3}d(f)$ in total to adjacent bad 3-faces by R6. Hence, we have $ch^*(f) \geq 2 \times d(f) - 10 - \frac{1}{3}d(f) \geq 0$.

The proof of Theorem 3.5 is completed. □

4 3-choosability

In 1995, Thomassen [6] proved that every planar graph of girth at least 5 is 3-choosable. And then in 2003, he [7] gave a shorter proof of this result. We find out that the argument used in [7] also works for signed graphs. Hence the following statement is true.

Theorem 4.1. *Every signed planar graph with neither 3-circuit nor 4-circuit is 3-choosable.*

For the sake of completeness, the proof is given in the appendix.

Theorem 4.2. *There exists a signed planar graph (G, σ) such that G has girth 4 and (G, σ) is not 3-choosable but G is 3-choosable.*

Proof. Let T be a plane graph consisting of two circuits $[ABCD]$ and $[MNPQ]$ of length 4 and four other edges AM, BN, CP and DQ, as shown in Figure 3. Take nine copies $T_0, \ldots, T_8$ of T, and identify $A_0, \ldots, A_8$ into a vertex A' and $C_0, \ldots, C_8$ into a vertex C'. Let G be the resulting graph. Clearly, G is planar and has girth 4.

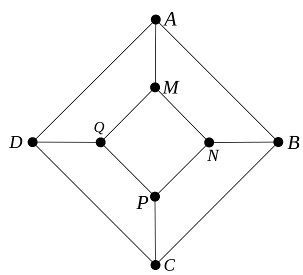

Figure 3: graph T

Define a signature σ of G as: $\sigma(e) = -1$ for $e \in \{M_iN_i : i \in \{0, \ldots, 8\}\}$, and $\sigma(e) = 1$ for $e \in E(G) \setminus \{M_iN_i : i \in \{0, \ldots, 8\}\}$.

For $i \in \{0, 1, 2\}$, let $a_i = i$ and $b_i = i + 3$. Define a 3-list-assignment L of G as follows: $L(A') = \{a_1, a_2, a_3\}, L(C') = \{b_1, b_2, b_3\}$; for $i, j \in \{0, 1, 2\}$, let $L(B_{3i+j}) = L(D_{3i+j}) = \{a_i, b_j, 6\}, L(N_{3i+j}) = L(Q_{3i+j}) = \{6, 7, -7\}, L(M_{3i+j}) = \{a_i, 7, -7\}$ and $L(P_{3i+j}) = \{b_j, 7, -7\}$.

We claim that signed graph (G, σ) has no L-coloring. Suppose to the contrary that c is an L-coloring of (G, σ). Let $c(A') = a_p$ and $c(C') = b_q$. Consider subgraph T_{3p+q}. It follows that $c(B_{3p+q}) = c(D_{3p+q}) = 6$. Furthermore, the circuit $[M_{3p+q}N_{3p+q}P_{3p+q}Q_{3p+q}]$ is unbalanced and thus not 2-choosable. Hence, T_{3p+q} is not properly colored in c, a contradiction. This proves that (G, σ) has no L-coloring and therefore, (G, σ) is not 3-choosable.

We claim that graph G is 3-choosable. For any 3-list-assignment of G, choose any color for vertices A' and C' from their color lists, respectively. Consider each subgraph T_i ($i \in \{0, \ldots, 8\}$). Both vertices B_i and D_i can be list colored. The 2-choosability of

circuit $[M_iN_iP_iQ_i]$ yields a list coloring of T_i and hence a list coloring of G. This proves that G is 3-choosable. $\square$

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5 Appendix

Theorem 5.1. *Let (G, σ) be a signed plane graph of girth at least 5, and D be the outer face boundary of G. Let P be a path or circuit of G such that $|V(P)| \leq 6$ and $V(P) \subseteq V(D)$, and σ_p be the restriction of σ to P. Assume that (P, σ_p) has a 3-coloring c. Let L be a list-assignment of G such that $L(v) = \{c(v)\}$ if $v \in V(P)$, $|L(v)| \geq 2$ if $v \in V(D) \setminus V(P)$, and $|L(v)| \geq 3$ if $v \in V(G) \setminus V(D)$. Assume furthermore that there is no edge joining vertices whose lists have at most two colors except for the edges in P. Then c can be extended to an L-coloring of (G, σ).*

Proof. We prove Theorem 5.1 by induction on the number of vertices. We assume that (G, σ) is a smallest counterexample and shall get a contradiction.

Claim 5.1.1. *G is 2-connected and hence, D is a circuit.*

We may assume that G is connected, since otherwise we apply the induction hypothesis to every connected component of G. Similarly, G has no cutvertex in P. Moreover, G has no cutvertex at all. Suppose to the contrary that u is a cutvertex contained in an endblock B disjoint from P. We first apply the induction hypothesis to $G-(B-u)$. If B has vertices with only two available colors joined to u, then we color each such vertex. These colored vertices of B together with the edges joining them to u divide B into parts each of which has at most three colored vertices inducing a path. Now we apply the induction hypothesis to each of those parts. This contradiction proves Claim 5.1.1.

Claim 5.1.2. *For $e \in E(P)$, e is not a chord of D.*

If some edge e of P is a chord of G, then e divides G into two parts, and we apply the induction hypothesis to each of those two parts. This contradiction proves Claim 5.1.2.

By Claims 5.1.1 and 5.1.2, we may choose the notion such that $D = [v_1 \dots v_k]$ and $P = v_1 \dots v_q$.

Let X be a set of colored vertices of G. To save writing we just say "delete the product colors of X from G" instead of "for $v \in V(G) \setminus X$, delete all of the colors in $\{c(u)\sigma(uv)\colon\ u \in X \text{ and } uv \in E(G)\}$ from the list of v".

Claim 5.1.3. *P is a path, and $q + 3 \leq k$.*

If $P = D$, then we delete any vertex from D, and delete the product color of that vertex from G. If $P \neq D$ and $k < q + 3$, then we color the vertices of D not in P, we delete them together with their product colors from G.

Now we apply the induction hypothesis to the resulting graph G', if possible. As G has grith at least 5, the vertices with precisely two available colors are independent. For the same reason, such a vertex cannot be joined to two vertices of P. However, such a vertex

may be joined to precisely one vertex of P. We then color it. Now the colored vertices of G' divide G' into parts each of which has at most 6 precolored vertices inducing a path. We then apply induction hypothesis to each of those parts. This contradiction proves Claim 5.1.3.

Claim 5.1.4. *D has no chord.*

Suppose to the contrary that xy is a chord of D. Then xy divides G into two graphs G_1, G_2, say. We may choose the notation such that G_2 has no more vertices of P than G_1 has, and subject to that condition, $|V(G_2)|$ is minimum. We apply the induction hypothesis first to G_1. In particular, x and y receive a color. The minimality of G_2 implies that the outer cycle of G_2 is chordless. So G_2 has at most two vertices which have only two available colors and which are joined to one of x and y. We color any such vertex, and then we apply the induction hypothesis to G_2. This contradiction proves Claim 5.1.4.

Claim 5.1.5. *G has no path of the form v_iuv_j where u lies inside D, except possibly when $q = 6$ and the path is of the form v_4uv_7 or v_3uv_k. In particular, u has only two neighbors on D.*

We define G_1 and G_2 as in the proof of Claim 5.1.4. We apply the induction hypothesis first to G_1. Although u may be joined to several vertices with only two available colors, the minimality of G_2 implies that no such vertex is in $G_2 - \{u, v_i, v_j\}$. There may be one or two vertices in $G_2 - \{u, v_i, v_j\}$ that have only two available colors and which are joined to one of v_i and v_j. We color any such vertex, and then at most six vertices of G_2 are colored. If possible, we apply the induction hypothesis to G_2. This is possible unless the coloring of G_1 is not valid in G_2. This happens only if P has a vertex in G_2 joined to one of v_i and v_j. This happens only if we have one of the two exceptional cases described in Claim 5.1.5.

Claim 5.1.6. *G has no path of the form v_iuwv_j such that u and w lie inside D, and $|L(v_i)| = 2$. Also, G has no path v_iuwv_j such that u and w lie inside D, $|L(v_i)| = 3$, and $j \in \{1, q\}$.*

Repeating the arguments in Claims 5.1.4 and 5.1.5, we can easily get Claim 5.1.6.

Claim 5.1.7. *If C is a circuit of G distinct from D and of length at most 6, then the interior of C is empty.*

Otherwise, we can apply the induction hypothesis first to C and its exterior and then to C and its interior. This contradiction proves Claim 5.1.7.

If $|L(v_{q+2})| \geq 3$, then we complete the proof by deleting v_q and its product color from G, and apply the induction hypothesis to $G - v_q$ and obtain thereby a contradiction. So we assume $|L(v_{q+2})| \leq 2$. By Claim 5.1.3, $|L(v_{q+2})| = 2$ and thus $|L(v_{q+3})| \geq 3$. If

$|L(v_{q+4})| \geq 3$, then we first color v_{q+2} and v_{q+1}, then we delete them and their product colors from G. We obtain a contradiction by applying the induction hypothesis to the resulting graph. By Claims 5.1.4 and 5.1.5 this is possible unless $q = 6$ and G has a vertex u inside D joined to both v_4 and v_7. In this case we color u and delete both v_5 and v_6 before we apply the induction hypothesis. Hence, we may assume that $|L(v_{q+4})| \leq 2$.

We give v_{q+3} a color not in $\{\alpha\sigma(v_{q+3}v_{q+4})\colon\ \alpha \in L(v_{q+4})\}$ and then color v_{q+2} and v_{q+1}, and finally we delete v_i and the product color of v_i from G for $i \in \{q+1, q+2, q+3\}$. We obtain a contradiction by applying the induction hypothesis to the resulting graph. If $q = 6$ and G has a vertex u inside D joined to v_4 and v_7, then, as above, we color u and delete v_5 and v_6 before we use induction. If $q = 6, q + 3 = k$, and G has a vertex u' inside D joined to v_3 and v_k, then we also color u' and delete v_1 and v_2 before we use induction. Finally, there may be a path $v_{q+1}wzv_{q+3}$ where w and z lies inside D. By Claim 5.1.7, this path is unique. We color w and z and delete them together with their product colors from G before we use induction. Note that u and u' may also exist in this case. If there are vertices joined to two colored vertices, then we also color these vertices before we use induction.

The colored vertices divide G into parts, and we shall show that each part satisfies the induction hypothesis. By second statement of Claim 5.1.6, there are at most six precolored vertices in each part, and they induce a path. Claim 5.1.5 and the first statement of Claim 5.1.6 imply that there is no vertex with precisely two available colors on D which is joined to a vertex inside D whose list has only two available colors after the additional coloring. Since G has girth at least 5 and by Claim 5.1.7, there is no other possibility for two adjacent vertices z and z' to have only two available colors in their lists, as both z and z' must be adjacent to a vertex that has been colored and deleted.

This contradiction completes the proof. $\square$